4401. (a) Raise the parametric equations to the power 4, and use the first Pythagorean trig identity.

(b) Consider that, for the square root to be well defined, $\cos t \in [0, 1]$.

(c) Compare the parametric equations in part (a) to the standard parametric equations of a unit circle, using the inequality in (b) and another equivalent one for $\sqrt{\sin t}$.

4402. (a) Consider $\alpha = \frac{b}{3a}$.

(b) Consider the cubic as a translation of the curve $y = px^3 + qx$.

4403. Establish that the points of inflection are at ± 1. So, the chord is over the domain $[-1, 1]$.

Find the area between the curve and the x axis, over $[-1, 1]$, using the normal distribution facility on a calculator (area is equivalent to probability).

Then subtract the area of a rectangle.

4404. Split the integrand into the sum of an integer and a proper algebraic fraction. Then write the latter in partial fractions.

4405. (a) P has no points of inflection, so $\text{f}''(x)$ does not change sign.

(b) Differentiate $\text{h}(x) = \text{f}(x) - mx - c$ twice.

(c) Consider $\text{h}(x) = 0$ as an equation for intercepts of P and $y = mx + c$.

(d) Q has non-negative curvature, so any chord lies at or above Q. The x axis is the chord in question.

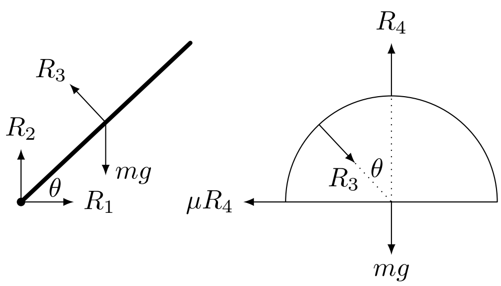
(e) Show that $\text{h}''(b) < 0$.

4406. This is not true. Consider a possibility space of eight equally likely outcomes

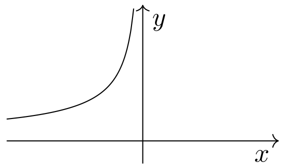
$$\{1, 2, 3, 4, 5, 6, 7, 8\}.$$

Look for a counterexample in which each of X, Y, Z contains four outcomes.

4407. Consider the case of limiting friction, in which the objects are in equilibrium, with friction at F_{max}. The force diagrams are



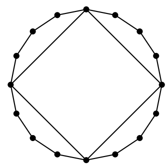
4408. The first equation is a quadratic. Factorise it, and use the factor theorem to show that, graphically, it consists of two separate sections. One is



$xy^{\frac{3}{2}} = -1$

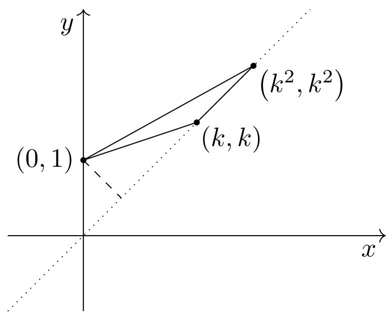
Sketch the other, and locate the point of tangency on one of the sections. (You don't need to find its coordinates, but you could do.) Show that the straight line must then intersect the other section exactly once.

4409. The scenario, with a successful outcome shown, is



4410. Make $\tan t$ the subject of the second equation. Use a double-angle formula on the first, square it, and divide through by $\sin^2 t \cos^2 t$. Replace $\operatorname{cosec}^2 t$ and $\sec^2 t$ using Pythagorean trig identities, and then substitute for $\tan t$.

4411. Use the dashed "height" below:



4412. Consider the quantity $S = x + y + z$.

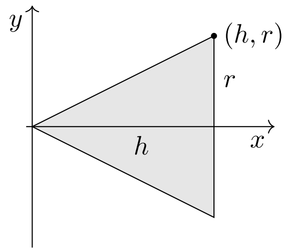
4413. (a) Use the reverse chain rule.

(b) Use the generalised binomial expansion, first taking out a factor of $9^{\frac{3}{2}}$.

(c) Find the initial velocity using $v = \sqrt{2t + 9}$. Then use the constant acceleration formula $s = ut + \frac{1}{2}at^2$ with unknown acceleration a. Show that a particular value of a produces the approximation in part (b).

4414. Establish that rotation clockwise by $90°$ around the origin is the same as reflecting in $y = 0$ and then $y = x$.

4415. Place the apex of the cone at the origin, and the base at $x = h$. In cross-section, the cone is then



Find the equation $y = ...$ of the upper slanted edge, then integrate πy^2 across the horizontal height of the cone.

4416. The boundary equations are $y = \pm \sin x$. Points in the solution set have y values closer to zero than points on these sinusoids.

4417. (a) Write each logarithm over base 64, by raising base and input to the same power.

(b) Use a sign change method.

4418. With friction limiting, the system is in equilibrium, and the tensions differ by exactly μR Newtons. You can work out the tensions, since the blocks aren't moving. Add these for the total downwards force on the pulley. Then set up an equation for the difference in tensions.

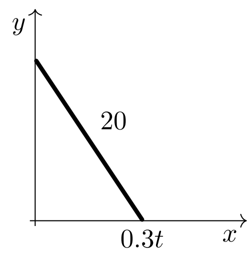
4419. Use the identity $\cos x \equiv \sin\left(\frac{\pi}{2} - x\right)$. Then consider harmonic form.

4420. (a) Differentiate implicitly and set $\frac{dy}{dx} = 0$ without rearranging. Substitute the result back into the original equation.

(b) Rearrange to make y the subject.

(c) The curve has odd symmetry.

4421. The scenario, with the ground as the x axis and the wall as the y axis, is as follows. Assuming that the foot of the ladder is at the origin at time $t = 0$, the x coordinate of A is given by $x = 0.3t$.



Write the y coordinate of B in terms of t, and differentiate to find its velocity.

4422. Write the integrand in partial fractions.

4423. (a) Substitute into the LHS and simplify.

(b) Use the parametric definitions. That the lines $x = \pm 2$ are tangent to the curve is immediate. To analyse $y = \pm 2$, consider harmonic form.

4424. (a) Differentiate by the chain rule.

(b) Differentiate again by the product rule.

(c) Consider the multiplicities of the roots of the second derivative, so as to show a sign change.

4425. Complete the square in both circles. Work out the set of possible values for the radius of the second circle. Consider the two boundary cases.

4426. A conditioning approach is tricky when the scores have been named following the experiment. So, use a combinatorics approach. There are $6^3 = 216$ outcomes in the possibility space. List successful (X, Y, Z) events systematically, then consider how many outcomes each such event represents.

4427. Use the substitution $x = \tan\theta$.

4428. (a) Consider the circle geometry of the bottom two bottles.

(b) Since all contacts are smooth, the only force supporting bottles $B_{k+1}, ..., B_n$ is the reaction force applied by bottle B_k. Taking bottles $B_{k+1}, ..., B_n$ as a one object, consider vertical equilibrium.

4429. Rearrange to $y^2 = 1/x - x^2$. Then consider instead $y = 1/x - x^2$. Sketch this as a sum of $y = 1/x$ and $y = x^2$. Then look back at the original graph.

4430. Solve the first equation as a cubic in $(a + 2b)$.

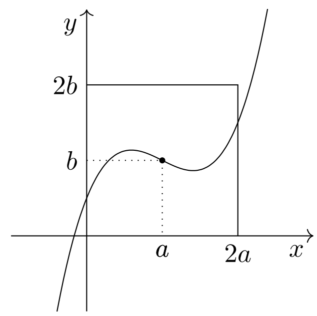
4431. (a) Set $y = 0$ and show that the resulting equation has a double root.

(b) Set $y = 0.4$ and do likewise.

(c) By symmetry, the centre of the circle must be at the midpoint of the points in (a) and (b).

4432. It is true. No solution is required.

4433. Consider the following diagram:



4434. Call the integral I, and use parts twice. First time around, let $u = \sin x$. Second time around, let $u = \cos x$. You want an equation of the form

$$I = \mathrm{f}(x) + \tfrac{1}{16}I.$$

At this point you can rearrange for I.

4435. Exponentiate each side of the proposed log law over base a. Then use the index law to show the two sides are the same.

4436. (a) The distribution is binomial.
(b) Consider negative values of $X_1 - X_2$.

4437. The plan is: show LHS and RHS are equivalent to

$$\frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}.$$

Use compound-angle formulae on $\tan\left(\frac{\pi}{3} \pm x\right)$, and simplify the resulting expressions. Then substitute into the RHS.

On the LHS, write $3x \equiv 2x + x$ and then use a compound-angle formula on $\tan(2x + x)$.

4438. Noting that $\mathrm{f}(a)$, $\mathrm{f}'(a)$ and $\mathrm{f}''(a)$ are all constant, differentiate the function $\mathrm{g}(t)$ twice with respect to t. Use the chain rule. Substitute $t = a$ into each.

4439. Assuming that θ is a small angle in radians, you can use both a small-angle approximation and the generalised binomial expansion.

4440. (a) Look for roots of the denominator.
(b) Integrate by inspection. Note the numerator is almost the derivative of $1 - x^2$.
(c) Set up an integral between $x = 0$ and $x = p$. Take the limit as $p \to 1$.

4441. Assume, for a contradiction, that the sequence is a GP. The common ratio is b/a. Equate this to the ratio of the second and third terms, and then the ratio of the third and fourth terms. You'll get a pair of simultaneous equations in a and b. Solve to show that $a = b = 0$ is the only solution. Rule this out.

4442. (a) Using the factor theorem, give explicit forms for the functions. Solve the relevant equation.
(b) Factorise. For the proof that the two roots are distinct, use the AM-GM inequality.

4443. The variables x and y only appear as x^2 and y^2. Hence, both curves have the x axis and the y axis as lines of symmetry. So, you need only consider the positive quadrant. Solve simultaneously to show that there are exactly two intersections for $x, y \geq 0$.

4444. (a) The parametric integration formula is

$$I = \int_{x_1}^{x_2} y\,dx = \int_{t_1}^{t_2} y\frac{dx}{dt}\,dt.$$

(b) For $t \in [0, 1)$, $\frac{dx}{dt} > 0$; while for $t \in (1, 2]$, $\frac{dx}{dt} < 0$. Interpret this fact graphically.
(c) Integrate by parts.

4445. Working with a tetrahedron of side length 1, find the perpendicular height of the base, and then the distance of the centre of the base from its vertices. Form a right-angled triangle and find the height. Scale up by length l, equate this to 2 and solve.

4446. Simplify the equation of the curve. You should find a parabola of the form $x = \mathrm{f}(y)$. Sketch the region and set up an integral with respect to y.

4447. Take out a factor of $(x^2 + y)$.

4448. Put everything onto the LHS and factorise it. Be careful with the direction of the implications in your argument.

4449. Find the equation of a generic tangent at $x = p$, and look for re-intersections with the curve.

4450. This is mainly a question of interpreting notation. The key thing is the definition of the index r: for k variables to change, the number available for change must be in $\{k, k + 1, ..., n\}$. We express this as integer values of r from $r = k$ to $r = n$. Once r is chosen, the distribution of the number of variables that do change is $Y \sim B(r, 1/2)$.

4451. Heron's formula gives $A^2 = s(s-a)^2(s-b)$. Express b in terms of s and a, and then set the derivative $\frac{d}{da}(A^2)$ to zero for optimisation.

4452. Consider the case $k = 0$.

4453. Find the equation of the dashed line, and then the coordinates of the point of intersection of the parabolae. Then set up a definite integral to find the area enclosed by $y = x^2$ and the line.

4454. The locus is a square.

4455. The first four terms of the proposed Cartesian equation may be factorised: consider them as the output of a binomial expansion.

4456. Find the derivatives. Sub them into the DE and simplify. Note: if a trial solution is to satisfy the DE, it must do so for *all values* of x.

4457. Write in partial fractions, then use the generalised binomial expansion.

4458. Integrate by substitution, with $u = 1 + e^{2x}$.

4459. Reflection in the line $x = k$ is equivalent to

- reflection in $x = 0$, then
- translation by vector $2k\mathbf{i}$.

Rotation by 90° anticlockwise around the origin is equivalent to

- reflection in $x = 0$, then
- then reflection in $y = x$.

4460. Find the average speed over the first k seconds by definite integration. Set your result equal to the instantaneous speed and simplify.

4461. Integrate by parts. When evaluating at $x = 0$, $\ln x$ is undefined, but the limit is nonetheless finite. Check a small value of x to ascertain its value. Solve the resulting equation by factorising. You should get $k = 0$, and another non-trivial value.

4462. Rewrite the sum as follows:

$$\begin{aligned}\sum_{r=1}^{n}(2r-1)^2 &\equiv \sum_{r=1}^{2n} r^2 - \sum_{r=1}^{n}(2r)^2 \\ &\equiv \sum_{r=1}^{2n} r^2 - 4\sum_{r=1}^{n} r^2.\end{aligned}$$

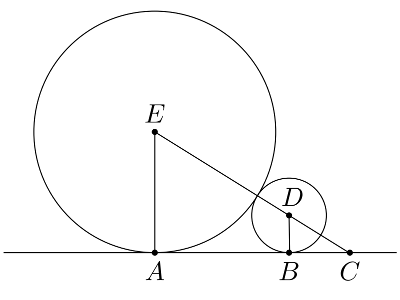
4463. Assume, for a contradiction, that $x = p/q$, where $p, q \in \mathbb{Z}$, and $2^{x+1} - 3^{x-1} = 0$. This gives

$$\begin{aligned} 2^{\frac{p}{q}+1} &= 3^{\frac{p}{q}-1} \\ \Longrightarrow 2^{\frac{p+q}{q}} &= 3^{\frac{p-q}{q}} \\ \Longrightarrow 2^{p+q} &= 3^{p-q}. \end{aligned}$$

Find a contradiction from here.

4464. In each case, consider all possible degrees of the polynomial $\mathrm{f}(x) - \mathrm{g}(x)$. The parity of the degree is the relevant fact. Note that, if $\mathrm{f}(x)$ and $\mathrm{g}(x)$ have the same degree and the same leading coefficient, then $\mathrm{f}(x) - \mathrm{g}(x)$ has a lower degree.

4465. Use similar triangles in the following diagram:



4466. Fill in the following table of periods, and then find lowest common multiples:

$\sin x$	2π	$\sin 4x$	$\frac{\pi}{2}$
$\cos x$	2π	$\cos 5x$	
$\tan x$	π	$\tan 6x$	

4467. Using the product rule, differentiate three times (implicitly), simplifying each time.

4468. You need to be careful here: you can't just equate the two ordinal formulae and solve, as the value of n could be different in the different formulae. Consider prime factorisation.

4469. (a) Consider the symmetry of sin and cos, as in the identity $\cos x \equiv \sin\left(\frac{\pi}{2} - x\right)$. Also, consider the value that maximises $\sin x + \cos x$.

(b) Evaluate $\mathrm{f}(x)$ at its maximum and minimum. Establish that these are greater than and less than 4. Then use the periodic nature of sine and cosine.

4470. (a) Substitute the parametric definitions into the LHS of the Cartesian equation.

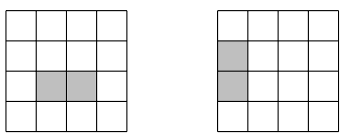
(b) Consider only the positive quadrant. Set up a definite integral with the parametric formula. Use a calculator to do the integral, having taken out a^2. Then take out a factor of π (divide by it on your calculator) to find the relevant rational k.

4471. (a) Factorise the first and second derivatives.

(b) The curve is a positive quartic with a triple root at $x = 0$ and a single root at $x = 4$.

(c) In the vicinity of $x = 0$, the quartic term can be neglected in favour of the other two. This gives an approximate cubic. In the vicinity of $x = 4$, the linear term can be neglected in favour of the other two.

4472. There are 16! ways of placing the numbers. You need 1 and 2 to form a rectangle, either vertically or horizontally.



Work out the number of different locations for the rectangle, and the number of ways of placing the numbers once the rectangle has been chosen.

4473. Integrate by parts twice. The tabular integration method is quicker.

4474. (a) The relevant fact is that, for a function to be self-inverse, $\mathrm{f}^2 : x \longmapsto x$.
(b) Use the same idea as in (a), now considering the degrees of the polynomials $\mathrm{f}(x)$ and $\mathrm{f}^2(x)$.

4475. (a) Consider the exponential decay of e^{-kt}, and the fact that it dominates the linear growth of $\left(t + \frac{1}{k}\right)$.
(b) Differentiate by the product rule.
(c) Use the results of (a) and (b).
(d) Set $\frac{da}{dt} = 0$.

4476. Write the RHS in harmonic form. Consider first the stationary values of the RHS. For this, you don't need to use calculus.

——— Alternative Method ———

Write the RHS in harmonic form. Then square both sides, differentiate and set the first derivative to zero. This should give four values, of which three produce well defined stationary points.

4477. The implication only goes one way. To set up a counterexample to the other direction, consider a constant of integration.

4478. (a) An even graph $y = \mathrm{f}(x)$ has the y axis as a line of symmetry: test the input $-x$ to show that $\mathrm{A}(-x) = \mathrm{A}(x)$.
An odd graph $y = \mathrm{f}(x)$ has $(0, 0)$ as a centre of rotational symmetry: test the input $-x$ to show that $\mathrm{B}(-x) = -\mathrm{B}(x)$.
(b) Consider $\frac{1}{2}\mathrm{A}(x) + \frac{1}{2}\mathrm{B}(x)$.

4479. Use a small-angle approximation for $\cos\theta$, and the generalised binomial expansion.

4480. (a) Solve simultaneously, eliminating y from the curves to produce the equation for fixed points of the given iteration.
(b) You can find another fixed-point iteration. However, as generally, the Newton-Raphson method is far more reliable.
(c) In a fixed-point iteration $x_{n+1} = \mathrm{g}(x_n)$, the condition for convergence is $|\mathrm{g}'(\alpha)| < 1$.

4481. Solve for x intercepts (one in terms of k). Sub these values into $\frac{dy}{dx} = 0$ and solve for k. Consider carefully the validity of any solutions.

4482. In (a) and (b), you can work out exactly what the solution set is. In (c), you can't. However, you can still work out the answer to the question.

4483. There are counterexamples with $q = -p$, which makes the proposed line of symmetry the y axis.

4484. Factorise the first equation.

4485. (a) Integrate by parts, with $u = \ln x$ and $v' = 1$.
(b) Write the integrand in terms of $\ln x$, using the change of base formula or similar. Then use the result from (a), and convert back to $\log_2 x$.

4486. From the graph, the range is $(-\infty, 0) \cup [k, \infty)$, where k is the y value of the local minimum.

4487. Statement (b) is true.

4488. Integrate by inspection and set up an equation in k. It isn't analytically solvable. Solve numerically (you'll find an exact root) and then justify why your value is the only possible value of k.

4489. Consider the points of the locus which lie on the lines $3x - y = 0$ and $x + y = 0$.

4490. Consider the graph $y = \mathrm{f}(x)$. Rewrite as

$$\int_a^b y\,dx + \int_c^d x\,dy = bd - ac,$$

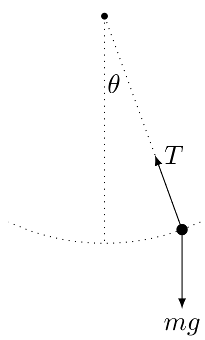
and consider bd and ac as the areas of rectangles.

4491. The only way the triangles don't overlap is if, taken in order around the circumference, A, B, C and D, E, F form distinct groups. The exact positions of the points aren't relevant, only their order is.

Consider the possibility space as an alphabetical list of the 6! ways of ordering the points.

4492. Let $z = \mathrm{f}(x)$ and $y = \mathrm{g}(x)$. Find and simplify $\frac{dz}{dy}$, using the chain rule. Call the result u. Then find and simplify $\frac{du}{dy}$, using chain and quotient rules.

4493. The force diagram is as follows:



Let x be the arc length/position, taken positively from the equilibrium position. With θ in radians, $x = l\theta$. Differentiate this twice with respect to t. Resolve in the tangential direction, and use a small-angle approximation.

4494. You don't need any calculus here: the radicand is a quadratic in $x^{-\frac{4}{5}}$.

4495. There are three possibilities: $Y_1 > Y_2$ or $Y_1 < Y_2$ or $Y_1 = Y_2$. The first two are symmetrical. So, calculate the third.

4496. Find the reaction forces, and thus the maximal frictional forces. Then assume that the system is in limiting friction, with both frictional forces at F_{max}. Consider equilibrium in the direction of the string, bypassing the tension.

4497. Show, using a double-angle formula, that

$$\sqrt{8} = \frac{2m}{1 - m^2}.$$

Solve this equation.

4498. The fact that determines whether there are values u_n which cannot be calculated with certainty is the parity of p and q.

4499. Square each of the variable equations, then add and subtract them. Combine your results to show that the equation is an ellipse in the (s, t) plane. Describe the (s, t) axes in the (x, y) plane.

4500. The quadrilateral is a trapezium.

——— End of 45th Hundred ———